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Characterisation of intermittency in chaotic systems

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Abstract. We discuss the characterisation of intermittency in chaotic dynamical systems by means of the time fluctuations of the response to a slight perturbation on the trajectory. A set of exponents is introduced which generalise the maximum Lyapunov characteristic exponent. The link with the statistical mechanics formalism is emphasised and we show that the exponents are connected to a free energy formally defined for chaotic systems. We perform some analytical computations in simple cases and give a few numerical examples.

1. Introduction

Intermittency is one of the most amazing phenomena in nonlinear dynamical systems. One observes ordered motion in phase space for long time, interrupted by randomly distributed bursts of strong chaoticity (Pomeau and Manneville 1980, Manneville and Pomeau 1980, Hirsch *et al* 1982). This strange behaviour was first observed in the experimental analysis of fully developed turbulence (Kuo and Corrsin 1971, Monin and Yaglom 1975). In this context we have recently proposed a model which describes the fractal structure of turbulent eddies in terms of local velocity fluctuations (Benzi *et al* 1984a) and a similar analysis was applied to strange attractors of chaotic systems using the moments of point density on the attractor (Paladin and Vulpiani 1984). In the spirit of this approach, time intermittent behaviour can be quantitatively characterised using the moments of the response function to perturbations (Fujisaka 1983, Benzi *et al* 1984b). A similar approach has been also introduced by Grassberger (1984) in order to study the relation between Lyapunov characteristic exponents and fractal dimension in chaotic systems.

On the other hand it has been pointed out that the formalism of statistical mechanics may be usefully applied to the theory of dynamical systems. Rigorous results were obtained by Bowen (1975, 1979), Ruelle (1976, 1978), Sinai (1968, 1972) and Walters (1976) mainly for axiom A systems. It is possible in this case to introduce the concept of Gibbs states, entropy, pressure and all the machinery of classical one-dimensional spin models. It is unfortunately hard to extend these results to more general cases but a formalism, appealing to intuition, has been introduced for chaotic systems (Oono and Takahashi 1980, Takahashi and Oono 1984).

We discuss the characterisation of intermittency in § 2 introducing a set of generalised Lyapunov exponents which are somehow analogous, for time behaviour, to the quantities introduced by Benzi *et al* (1984a). In § 3 we show the link of this approach with the statistical mechanics formalism on a rather naive level. Numerical results for some dynamical systems are exhibited in § 4.

2. Generalised Lyapunov exponents

Let us consider a dynamical system, i.e. either a set of differential equations or a map:

$$\dot{x} = f(x), \qquad x_n = g(x_{n-1}) \tag{1}$$

with $x, f, g \in \mathbb{R}^d$. The greatest Lyapunov exponent λ is defined as:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\zeta(t)\|}{\|\zeta(0)\|} \tag{2}$$

where ζ is the tangent vector whose evolution is given by:

$$\dot{\zeta}_i = \sum_{j=1}^{d} \frac{\partial f_i}{\partial x_j} \zeta_j \qquad i = 1, \ldots, d. \tag{3}$$

λ measures the sensitivity to initial conditions and is therefore an indicator of the degree of chaos (Benettin *et al* 1976, 1980a, b). Positive λ implies exponential error growth with characteristic time λ^{-1}

$$\delta x(t) \sim \delta x(0)\, \mathrm{e}^{\lambda t} \tag{4}$$

where $\delta x(0)$ is the initial error. Lyapunov exponents cannot describe the degree of intermittency for a chaotic dynamical system. In fact a mathematical definition of intermittency has not yet been given.

We can think of intermittency as a non-uniform distribution in time of 'chaotic behaviour'. Hence we wish to give a quantitative description for the fluctuations of the 'degree of chaos' in the system. To this purpose we introduce in this section a set of generalised Lyapunov exponents (Fujisaka 1983, Benzi *et al* 1984b) which could measure the intermittency.

Let $R_t(\tau)$ be the response function after a time τ of a perturbation at time t:

$$R_t(\tau) = \frac{\|\zeta(t+\tau)\|}{\|\zeta(t)\|}. \tag{5}$$

We define $L(q)$ as

$$\overline{R(\tau)^q} \sim \exp(L(q)\tau) \tag{6}$$

where $\overline{(\ldots)}$ is a time average and τ is large enough. $L(q)$ recalls respectively the quantities ζ_q and $\Phi(q)$ previously introduced (Benzi *et al* 1984a, Paladin and Vulpiani 1984) to characterise spatial intermittency in fully developed turbulence and the non-homogeneous structure of strange attractors.

Let us first consider the case without fluctuations of $R_t(\tau)$. This is the non-intermittent case. One has, in the absence of fluctuations,

$$L(q) = \lambda q. \tag{7}$$

It follows that deviations from (7) give a measure of the degree of intermittency. One can show that $L(q)$ is convex in q (Feller 1971). Moreover in the general case we have:

$$\lambda = \mathrm{d}L(q)/\mathrm{d}q|_{q=0}. \tag{8}$$

It is sometimes stated that a system is stable under small perturbations if $\lambda < 0$. However simple arguments indicate that if $\lim_{q\to\infty} L(q) > 0$ then the system has a finite probability that a small perturbation gives a large response. Let us give a simple example. We can compute analytically $L(q)$ in the case of the one-dimensional Langevin equation

$$\dot{x} = -\mathrm{d}V/\mathrm{d}x + \eta \tag{9}$$

where η is a white noise, i.e. a random Gaussian process with zero average and covariance:

$$\langle \eta(t)\eta(t')\rangle = \delta(t-t'). \tag{10}$$

We consider two trajectories $x(t)$ and $\tilde{x}(t) = x(t) + \varepsilon(t)$, both satisfying equation (9) with the same realisation of the noise. We define

$$R(x(0), \tau|\eta) = \lim_{\varepsilon(0)\to 0} \frac{|\varepsilon(\tau)|}{|\varepsilon(0)|} \tag{11}$$

which is of course a functional of η. The function $L(q)$ is defined as:

$$L(q) = \lim_{\tau\to\infty} \frac{1}{\tau} \ln \int \mathrm{d}[\eta] P[\eta] R^q(x(0), \tau|\eta) \tag{12}$$

where $P[\eta]$ is the Gaussian probability distribution functional. Usual arguments imply that $L(q)$ does not depend on $x(0)$. Our aim is to evaluate $L(q)$. We first notice that

$$R(x(0), \tau|\eta) = \exp\left(-\int_0^\tau V''(x(t))\,\mathrm{d}t\right) \tag{13}$$

where the dependence on η comes through the dependence of $x(t)$ on η. Let us evaluate

$$\overline{R^q(\tau)} = \int \mathrm{d}\mu[x]^\tau \exp\left(-q\int_0^\tau V''(x(t))\,\mathrm{d}t\right) \tag{14}$$

where $\mathrm{d}\mu[x]^\tau$ is the measure induced on the trajectories by the stochastic differential equation (9). It is well known that (see e.g. Graham 1978):

$$\begin{aligned} \mathrm{d}\mu[x]^\tau &= \mathrm{d}[x]^\tau \exp\left(-\int_0^\tau (\tfrac{1}{2}\dot{x}(t)^2 + U(x(t)))\,\mathrm{d}t\right) \\ &= \mathrm{d}P[x]^\tau \exp\left(-\int_0^\tau U(x(t))\,\mathrm{d}t\right) \end{aligned} \tag{15}$$

where $\mathrm{d}P[x]^\tau$ is the usual Wiener measure on the trajectories and we have neglected the boundary terms (i.e. terms depending on $x(0)$ and $x(\tau)$). The function U is given by:

$$U(x) = \tfrac{1}{2}(\mathrm{d}V/\mathrm{d}x)^2 - \tfrac{1}{2}\mathrm{d}^2V/\mathrm{d}x^2. \tag{16}$$

The Feynman path integral representation for quantum mechanics at imaginary time implies that (Feynman and Hibbs 1963):

$$\int \mathrm{d}\mu[x]^{\tau} \xrightarrow[\tau\to\infty]{} \exp(-\tau E_0(U)) \tag{17}$$

where we have neglected the prefactor and $E_0(U)$ is the ground state of the Hamiltonian

$$\hat{H} = -\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2 + U(x). \tag{18}$$

Consistency requires that $E_0(U) = 0$, which is indeed trivial to check. We can now write

$$\overline{R^q(\tau)} = \int \mathrm{d}[x]^{\tau} \exp\left(-\int_0^{\tau} (\tfrac{1}{2}\dot{x}(t)^2 + U(x(t)) + qV''(x(t)))\,\mathrm{d}t \right)$$

$$\sim \exp(-\tau E_0(q)) \tag{19}$$

where $E_0(q)$ is the ground state of the Hamiltonian

$$\hat{H}(q) = -\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2 + U(x) + qV''(x). \tag{20}$$

It is clear that

$$L(q) = -E_0(q). \tag{21}$$

Equation (21) gives us a method to compute the $L(q)$ in an approximate way; a few exact solutions are available: for example if $V = x^2$ we have $L(q) = -2q$. In the general case we remark that $\lim_{q\to\infty} L(q)/q = -L^*$ is easy to compute and it is given by $L^* = \min_x V''(x)$. This means that, although $L(q)$ can be negative for small q and the system is stable in the usual sense, the system can be unstable under a small perturbation as soon as $V''(x)$ is negative somewhere as previously discussed.

We remark that this situation is somewhat similar to what happens in the case of multifractal sets in both fully developed turbulence and chaotic attractors: the strongest singularities dominate the behaviour of the high moments for the structure functions (Benzi *et al* 1984a; for a similar case in a different context see Berry (1977, 1982)).

3. Statistical mechanics formalism and intermittency

Chaotic systems do generally not satisfy axiom A and the extension of the statistical mechanics formalism is questionable. Nevertheless some authors (see e.g. Takahashi and Oono 1984) proposed a formal generalisation for the case of one-dimensional maps. Let us show how by this approach we can achieve a deeper understanding of the generalised Lyapunov exponents. For a n-spins system with periodic boundary conditions the partition function Z_n is the sum on all configurations of length n of the Boltzmann weight $\mathrm{e}^{-\beta H_n}$, where H_n is the energy of the configuration and β^{-1} the temperature. The free energy for spin is given in the thermodynamical limit by:

$$F(\beta) = -\frac{1}{\beta} \lim_{n\to\infty} \left(\frac{1}{n} \ln Z_n \right). \tag{22}$$

Takahashi and Oono (1984) introduced a free energy $F(g, \beta)$ for unit time in the case of one-dimensional maps $x_{n+1} = g(x_n)$ via the relation (22), defining the partition function as

$$Z_n(g, \beta) = \sum_{y\in \mathrm{Fix}(g^n)} \exp(-\beta \ln|(g^n)'(y)|) \tag{23}$$

where $\mathrm{Fix}(g^n)$ is the set of the unstable fixed points of $g^n = g \circ g \circ \ldots \circ g$ (n times) and $'$ indicates derivation with respect to y. Relation (23) is the analogue of the partition function which has already been defined for axiom A systems (Ruelle 1978). Let us call $R_n(y)$ the response function at time n to a perturbation around y, unstable periodic solution of period n. Relation (23) becomes

$$Z_n(g, \beta) = \sum_{y \in \mathrm{Fix}(g^n)} \exp(-\beta \ln R_n(y)). \tag{24}$$

It seems reasonable to estimate (24) by a time average $\overline{(\cdot)}$ in the chaotic phase of the system, where ergodicity should hold. It is known that $F(g, \beta = 1) = 0$ for chaotic maps g supported by absolute continuous measures (Ledrapier 1981). It follows by (22) and (23) that the weight to be used in an 'ensemble' average is $|(g^n)'|^{-1}$. Thus one has

$$\overline{(\cdot)} = \sum_{y \in \mathrm{Fix}(g^n)} (\cdot) \frac{1}{|(g^n)'|}$$

and

$$Z_n(g, \beta) = \overline{R_n^{-(\beta-1)}}. \tag{25}$$

Comparing equations (6) and (25) we obtain

$$L(1-\beta) = -\beta F(g, \beta). \tag{26}$$

The greatest Lyapunov exponent λ as defined in (8) is related to the internal energy $V = (\partial/\partial\beta)(\beta F)$ at $\beta = 1$:

$$\lambda = V(g, \beta)|_{\beta=1}. \tag{27}$$

It is tempting to make an analogy between classical statistical mechanics of spin systems and the theory of dynamical systems by noting that the no-intermittency case corresponds to a constant internal energy, i.e. to the case of high temperature. Intermittency is allowed if the 'temperature' of the system is reduced and the thermodynamic functions are not trivial (for instance the internal energy is a function of the temperature). Let us finally remark that formula (26) allows, at least formally, the computation of a free energy for dynamical systems also in the case of dimension greater than one and for differential dynamical systems.

4. Discussions and numerical results

The functional dependence of $L(q)$ on q contains all the information on the fluctuations of $R_t(\tau)$. In the simple case that a central limit theorem can be applied to $\ln R_\tau(\tau)$, $L(q)$ has the form:

$$L(q) = \lambda q + \tfrac{1}{2}\mu q^2 \tag{28}$$

where μ is related to the variance of $\ln R(\tau)$:

$$\langle(\ln R(\tau) - \langle \ln R(\tau)\rangle)^2\rangle = \mu\tau. \tag{29}$$

In this case μ/λ is a parameter measuring the degree of intermittency: $\mu/\lambda = 1$ is the critical value which discriminates between weak and strong intermittent behaviour.

Indeed, the log-normal probability distribution that we have assumed

$$P(R(\tau))=\frac{1}{(2\pi\mu\tau)^{1/2}R(\tau)}\exp\left(-\frac{(\ln R(\tau)-\lambda\tau)^2}{2\mu\tau}\right) \tag{30}$$

has a maximum for

$$\tilde{R}(\tau)=\exp[\lambda\tau(1-\mu/\lambda)]. \tag{31}$$

It follows, in the large time limit, that:

$$\tilde{R}(\tau)\xrightarrow[\tau\to\infty]{}0 \qquad \mu/\lambda>1 \tag{32a}$$

$$\tilde{R}(\tau)\xrightarrow[\tau\to\infty]{}\infty \qquad \mu/\lambda<1. \tag{32b}$$

We obtain that corrections to the 'mean field' (i.e. taking into account only the maximum of the probability distribution) due to the intermittent behaviour cannot be neglected if $\mu/\lambda>1$, giving a quite different qualitative behaviour. The two cases are sketched in figure 1: in the case (32*b*) intermittent corrections change only the quantitative behaviour of $L(q)$ at variance with case (32*a*). We remark that in general a log-normal distribution for $R_t(\tau)$ cannot be assured as it is not usually true that thermodynamical functions have a universal functional relation with the temperature.

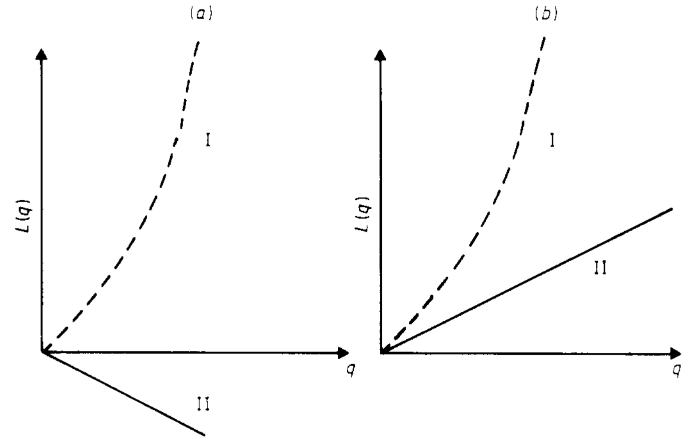

Figure 1. Schematic view of $L(q)$ for the log-normal distribution of $R(\tau)$. The full line (I) is $L(q)$ in the mean-field approximation, the broken curve (II) refers to equation (28). (*a*) $\mu/\lambda>1$. (*b*) $\mu/\lambda<1$.

To discuss this point further we have performed a series of numerical computations of $L(q)$ for the Henon–Heiles (1964) model, the Henon map (1976) and Lorenz system (1963) (see figures 2–4). The Henon–Heiles model shares a good consistency of $L(q)$ with equation (28) with μ/λ near to 1. On the contrary the Henon map and the Lorenz model for r near 166.07 (the critical value of intermittency transition to turbulence, see Manneville and Pomeau (1980)) shows strong deviations from a log-normal distribution of $R_t(\tau)$. We finally remark that in the Lorenz system for r near 24.74 (the critical value for the fixed points to be unstable) a linear behaviour of $L(q)$ is found.

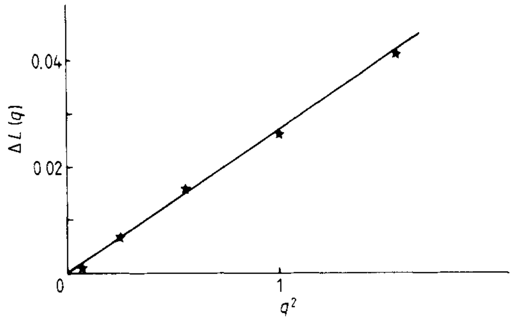

Figure 2. $\Delta L(q) = L(q) - \lambda q$ against q^2 for the Henon-Heiles model. The line indicates $\Delta L(q) = \frac{1}{2}\mu q^2$. The initial conditions on the chaotic region at the energy surface $E = 0.125$ (integration time 2×10^5).

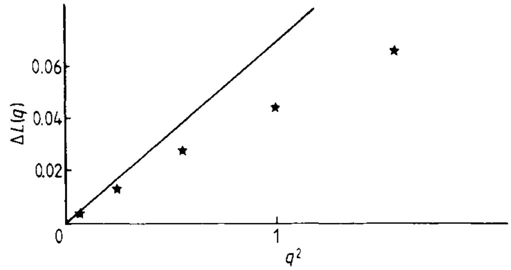

Figure 3. As figure 2 for the Henon map with $a = 1.2$, $b = 0.3$ (10^5 iterations).

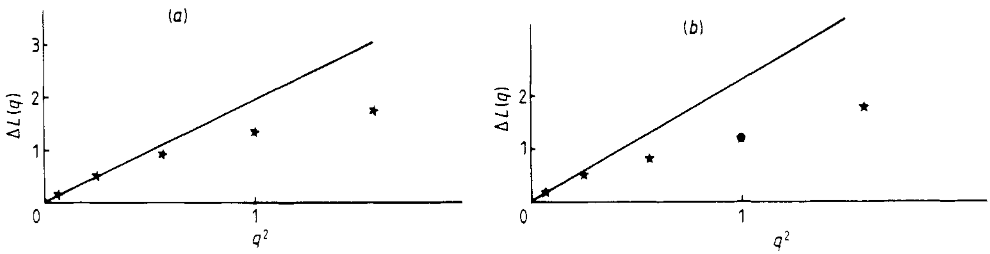

Figure 4. As figure 2 for the Lorenz model at (*a*) $r = 166.1$ and (*b*) $r = 166.3$ (integration time 10^5).

The last result can be understood with a simple analytical argument. It is known that the Lorenz model for $r \gtrsim 24.74$ can be qualitatively described (Lorenz 1963) by the one-dimensional map:

$$x_{n+1} = g(x_n) = \begin{cases} 2x_n & \text{for } x_n < \frac{1}{2} \\ 2(1 - x_n) & \text{for } x_n > \frac{1}{2}. \end{cases} \tag{33}$$

It is simple to compute the function $L(q)$ for the map (33) as defined by equations (22), (23) and (26). Because the number of the unstable fixed points for the n iteration is $2^n - 1$ with slope $\pm 2^n$, we obtain

$$Z_n = (2^n - 1)\exp(-\beta \ln 2^n),$$

$$F(\beta) = -\frac{1}{\beta}\lim_{n\to\infty}\left(\frac{1}{n}\ln Z_n\right) = -\frac{1}{\beta}(\ln 2)(1-\beta).$$

Hence we get $L(q) = q \ln 2$ which shows that map (33) has no intermittency with Lyapunov exponent $\ln 2$. Modifying the map (33) in order to have different slopes in different regions, we can easily obtain intermittent behaviour. For instance let us consider the following map:

$$x_{n+1} = g(x_n) = \begin{cases} 4\alpha x_n & \text{for } 0 \leq x_n \leq \frac{1}{4} \\ 4(1-\alpha)(x_n - \frac{1}{2}) + 1 & \frac{1}{4} \leq x_n \leq \frac{1}{2} \\ 4(\alpha - 1)(x_n - \frac{1}{2}) + 1 & \frac{1}{2} \leq x_n \leq \frac{3}{4} \\ 4\alpha(1 - x_n) & \frac{3}{4} \leq x_n \leq 1. \end{cases} \tag{34}$$

For $\alpha = \frac{1}{2}$ we recover the previous case (33). In order to show the intermittency for $\alpha \neq \frac{1}{2}$, we compute the quantity:

$$S = \overline{(\ln|g'|)^2}/\overline{\ln|g'|}^2 - 1 \tag{35}$$

which measures the fluctuations of the response as a function of α. We find that for α near 0.5 $S \sim (\alpha - \frac{1}{2})^2$ (see figure 5) which shows how the intermittency is built into the system.

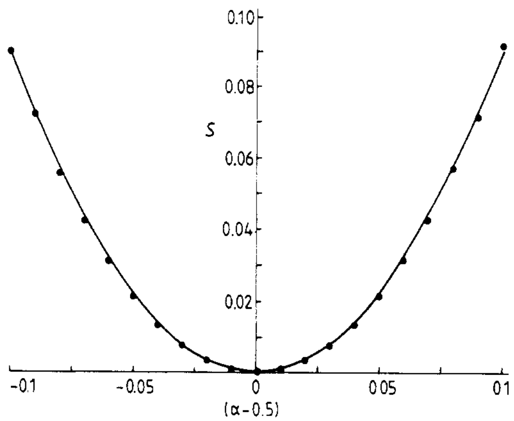

Figure 5. Plot of S against α. The full line indicates the curve $(\alpha - \frac{1}{2})^2$.

Acknowledgments

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